Assignment 1: Double Integrals: Solutions

1. (a) With $n = 4$ and $m = 2$, the eight lower-left points are

$$(0,0),(1/2,0),(1,0),(3/2,0),(0,1),(1/2,1),(1,1),(3/2,1)$$

Since each box has area $(1/2)(1) = 1/2$, the estimate for the integral is

$$(1/2)[f(0,0)+f(1/2,0)+f(1,0)+(3/2,0)+f(0,1)+f(1/2,1)+f(1,1)+f(3/2,1)]$$

which reduces to

$$(1/2)(0+0+0+0+0+1/4+1+9/4) = 7/4$$

(b) (Note: I made a mistake with this question - the curves given actually bound two different areas. For this solution we'll look at the bounded area on the right; both areas are identical. If you did the integral over both bounded areas, I will not deduct marks).

The curves $y = x^2 - 1$ and $y = 2$ intersect at $x = \sqrt{3}$. So we could take as our bounding rectangle $R = [0, \sqrt{3}] \times [0, 2]$. Dividing this rectangle up with $n = 4$ and $m = 2$, the lower left points are

$$(0,0),(\sqrt{3}/4,0),(\sqrt{3}/2,0),(3\sqrt{3}/4,0),(0,1),(\sqrt{3}/4,1),(\sqrt{3}/2,1),(3\sqrt{3}/4,1)$$

The area of each rectangle is $\sqrt{(3)/4}$. Of the 8 points, the first 5 give 0 when put into the function; the other three points are in the region, so we get

$$(\sqrt{3}/4)(3/16 + 3/4 + 27/16) = 21\sqrt{3}/32$$

2. (a)

$$\begin{aligned}
& \int_0^2 \int_0^3 x - y \, dx \, dy \\
= & \int_0^2 (x^2/2 - xy)|_0^3 \, dy \\
= & \int_0^2 (9/2 - 3y) \, dy \\
= & (9y/2 - 3y^2/2)|_0^2 \\
= & 9 - 6 \\
= & 3
\end{aligned}$$

(b) If we use Fubini's theorem to switch the order of integration, the integral is easier to calculate:

$$
\begin{aligned}
& \int_0^1 \int_1^2 y e^{xy} \, dy \, dx \\
= & \int_1^2 \int_0^1 y e^{xy} \, dx \, dy \text{ (by Fubini's theorem)} \\
= & \int_1^2 e^{xy}|_0^1 \, dy \\
= & \int_1^2 e^y - 1 \, dy \\
= & (e^y - y)|_1^2 \, dy \\
= & (e^2 - 2) - (e - 1) \\
= & e^2 - e - 1
\end{aligned}
$$

(c)

$$
\begin{aligned}
& \int_0^4 \int_y^{\sqrt{y}} x^3 + 4y \, dx \, dy \\
= & \int_0^4 (x^4/4 + 4xy)|_y^{\sqrt{y}} \, dy \\
= & \int_0^4 y^2/4 + 4y\sqrt{y} - y^4/4 - 4y^2 \, dy \\
= & \int_0^4 -15y^2/4 + 4y^{3/2} - y^4/4 \, dy \\
= & (-5y^3/4 + 8y^{5/2}/5 - y^5/20)|_0^4 \\
= & -80 + 256/5 - 256/5 \\
= & -80
\end{aligned}
$$

(d) The region D is type 2, with $0 \le y \le \pi/2$ and $-y \le x \le y$. So the integral is

$$
\begin{aligned}
& \int_0^{\pi/2} \int_{-y}^{y} \cos y \, dx \, dy \\
= & \int_0^{\pi/2} (x \cos y)|_{-y}^{y} \, dy \\
= & \int_0^{\pi/2} y \cos y - (-y \cos y) \, dy
\end{aligned}
$$

$$= \int_0^{\pi/2} 2y\cos y\,dy$$

To evaluate this integral, we use integration by parts with $u = y$ and $dv = \cos y$. Then the integral becomes

$$\begin{aligned}
&= 2\left(y\sin y|_0^{\pi/2} - \int_0^{\pi/2} \sin y\,dy\right) \\
&= 2\left(\pi/2 - (-\cos y)|_0^{\pi/2}\right) \\
&= 2\left(\pi/2 - 1\right) \\
&= \pi - 2
\end{aligned}$$

(e) When we change to polar co-ordinates, the region is bounded by $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$, while $1 - x^2$ becomes $1 - r^2\cos^2\theta$. So the integral is

$$\begin{aligned}
&\int_0^{2\pi}\int_0^2 (1 - r^2\cos^2\theta) r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^2 r - r^3\cos^2\theta\,dr\,d\theta \\
&= \int_0^{2\pi} (r^2/2 - r^4/4)\cos^2\theta)|_0^2\,d\theta \\
&= \int_0^{2\pi} 2 - 4\cos^2\theta\,d\theta
\end{aligned}$$

We now use the identity $\cos^2\theta = 1/2(1 + \cos(2\theta))$ to simplify the integral:

$$\begin{aligned}
&= \int_0^{2\pi} 2 - 2(1 + \cos(2\theta))\,d\theta \\
&= \int_0^{2\pi} \cos 2\theta\,d\theta \\
&= \sin(2\theta)/2|_0^{2\pi} \\
&= 0
\end{aligned}$$

3. The region is type 2, with $-\pi \leq y \leq \pi$ and $\sin y \leq x \leq e^y$. So the area is given by

$$\int_{-\pi}^{\pi}\int_{\sin y}^{e^y} 1\,dx\,dy$$

$$
\begin{aligned}
&= \int_{-\pi}^{\pi} e^y - \sin y \, dy \\
&= (e^y + \cos y)|_{-\pi}^{\pi} \\
&= (e^{\pi} - 1) - (e^{-\pi} - 1) \\
&= e^{\pi} - e^{-\pi}
\end{aligned}
$$

4. When drawn out, one can see that the curve is a four-leaf clover, with one leaf enclosed between $-\pi/4 \leq \theta \leq \pi/4$ (see Example 8 in Section 10.3 of the textbook). Thus, the area of the region is given by the integral

$$
\begin{aligned}
&\int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (1) r \, dr \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} (r^2/2)|_0^{\cos 2\theta} \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} 1/2(\cos^2(2\theta)) \, d\theta
\end{aligned}
$$

We now use the identity $\cos^2 \theta = 1/2(1 + \cos(2\theta))$ to simplify the integral:

$$
\begin{aligned}
&= \int_{-\pi/4}^{\pi/4} 1/4(1 + \cos(4\theta)) \, d\theta \\
&= 1/4(\theta + 1/4 \sin(4\theta))|_{-\pi/4}^{\pi/4} \\
&= 1/4[(\pi/4 + 0) - (-\pi/4 + 0)] \\
&= \pi/8
\end{aligned}
$$

5. The curves intersect at $x = 2$ and $x = -1$. The region is type 1, with $-1 \leq x \leq 2$ and $2 - x \leq y \leq 4 - x^2$, so that the volume is given by

$$
\begin{aligned}
&\int_{-1}^{2} \int_{2-x}^{4-x^2} x^2 + 4 \, dy \, dx \\
&= \int_{-1}^{2} (x^2 + 4)(4 - x^2 - (2 - x)) \, dx \\
&= \int_{-1}^{2} (x^2 + 4)(2 - x^2 + x) \, dx \\
&= \int_{-1}^{2} 2x^2 + 8 - x^4 - 4x^2 + x^3 + 4x \, dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-1}^{2} 8 + 4x - 2x^2 + x^3 - x^4 \, dx \\
&= (8x + 2x^2 - 2/3x^3 + 1/4x^4 - 1/5x^5)|_{-1}^{2} \\
&= (16 + 8 - 16/3 + 4 - 32/5) - (-8 + 2 + 2/3 + 1/4 + 1/5) \\
&= 423/20
\end{aligned}
$$

6. We can consider two parts of the solid seperately: the part above the xy-plane, and the part below. Since these regions are the same, the total volume is twice the volume of the part above the xy-plane. For the part above the xy-plane, the sphere meets the xy-plane with the equation $x^2 + y^2 = 25$. So, in the xy-plane, our region is bounded by $x^2 + y^2 = 25$ and $x^2 + y^2 = 9$. If we switch to polar co-ordinates, these are the curves with $r = 5$ and $r = 3$. Re-arranging the equation $x^2 + y^2 + z^2 = 25$ gives $z = \sqrt{25 - x^2 - y^2} = \sqrt{25 - r^2}$. Thus the total volume is

$$
\begin{aligned}
& 2 \int_{0}^{2\pi} \int_{3}^{5} (\sqrt{25 - r^2}) r \, dr \, d\theta \\
&= 2 \int_{0}^{2\pi} \int_{3}^{5} r(25 - r^2)^{1/2} \, dr \, d\theta \\
&= 2 \int_{0}^{2\pi} (-1/3(25 - r^2)^{3/2})|_{3}^{5} \, d\theta \\
&= 2 \int_{0}^{2\pi} (0 - (-64)/3) \, d\theta \\
&= 2(64\theta/3)|_{0}^{2\pi} \, d\theta \\
&= 2(64/3)(2\theta) \\
&= 256\pi/3
\end{aligned}
$$

7. The region is given by the bounds $0 \leq x \leq a$ and $0 \leq a - x$. To find the centre of mass, we first need to the total mass. This is given by the integral

$$
\begin{aligned}
& \int_{D} k(x^2 + y^2) \, dA \\
&= k \int_{0}^{a} \int_{0}^{a-x} x^2 + y^2 \, dy \, dx \\
&= k \int_{0}^{a} (x^2 y + y^3/3)|_{0}^{a-x} \, dx
\end{aligned}
$$

$$\begin{aligned}
&= k\int_0^a x^2(a-x) + (a-x)^3/3\,dx \\
&= k\int_0^a ax^2 - x^3 + (a-x)^3/3\,dx \\
&= k(ax^3/3 - x^4/4 - (a-x)^4/12)|_0^a \\
&= k((a^4/3 - a^4/4 - 0) - (0 - 0 - a^4/12)) \\
&= ka^4/6
\end{aligned}$$

To find the x co-ordinate of the centre of mass, we evaluate the integral

$$\int_0^a \int_a^{a-x} xk(x^2+y^2)\,dy\,dx$$

This is a similar integral to above, it evaluates to $ka^5/15$. Thus the x co-ordinate is given by

$$\frac{ka^5/15}{ka^4/6} = \frac{2a}{5}$$

Since the function and region is symmetric with regards to x and y, the y co-ordinate is also given by

$$\frac{2a}{5}$$

Thus the centre of mass is

$$\left(\frac{2a}{5}, \frac{2a}{5}\right)$$